family of solutions for the oscillatory case $h<2$ (the aperiodic case is simpler and will not be considered). We construct the matrix $Y(t)$ using the matrix $Y_2(t)$ of (2.17) taking the real and imaginary part of the first column of $Y_2(t)$ as its first and second column. After simple reduction we obtain the following asymptotic representations for the solutions of (4.1) ($b_{1,2}$ are the coordinates of the vector c):

$$q(t)=\left(b_1\cos\int_{t_0}^{t}\eta_1(\tau)\,d\tau+b_2\sin\int_{t_0}^{t}\eta_1(\tau)\,d\tau+o(1)\right)\exp\int_{t_0}^{t}\nu_1(\tau)\,d\tau$$

$$p(t)=\left[(b_1\nu_1(t)+b_2\eta_1(t))\cos\int_{t_0}^{t}\eta_1(\tau)\,d\tau+(b_2\nu_1(t)-b_1\eta_1(t))\sin\int_{t_0}^{t}\eta_1(\tau)\,d\tau+o(1)\right]\exp\int_{t_0}^{t}\nu_1(\tau)\,d\tau$$

$$\nu_1(t)=\operatorname{Re}\lambda_1(t)=-h(t)/2,\ \eta_1(t)=\operatorname{Im}\lambda_1(t)=(1-h^2/4)^{1/2}$$

Note 3. The above method of construcing the real *FSM* $Y(t)$ of the linearized system (2.2) using the *FSM* of $Y_2(t)$, with the asymptotic form given by (2.17), can also be used in the general case of n degrees of freedom. If the roots $\lambda_{i,n+i}$ are not real (when $\lambda_i=\overline{\lambda}_{n+i}$), then we take the real and imaginary part of the i-th column of $Y_2(t)$ as the i-th and $(n+i)$-th column of $Y(t)$. On the other hand, if $\lambda_{i,n+i}$ are real, then we take as the i-th and $(n+i)$-th column of $Y(t)$ the corresponding columns of the matrix $1/2\,(Y_2(t)+\overline{Y}_2(t))$. The fact that this yields a real *FSM* of $Y(t)$ of the linearized system (2.2) can be confirmed using the asymptotic representation (2.17) for $Y_2(t)$.

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ON TWO TYPES OF SWIRLING GAS FLOWS*

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Two classes of exact solutions of the three-dimensional stationary equations of gas dynamics are constructed. The solutions are used to describe isentropic gas flows with two types of swirling in axisymmetric divergent channels. The effect of swirling on the thrust of special-type nozzles is studied.

Approximate analytic or numerical methods were used earlier to study radial-equilibrium flows with arbitrary swirling in /1/, and various qualitative features of the swirling flows, such as the appearance of vacuum kernels, back flows and stagnation zones at the inlet to the nozzle throat were discussed in /2–4/. Analytic solutions in the transonic approximation were constructed in /5/ and the dependence of the nozzle thrust on the swirling parameters were investigated in /1, 6–9/.

1. In studying swirling gas flows we will use two classes of solutions of the equations of gas dynamics for the case when the velocity vector components u_i and the function $Q=\rho^{\gamma-1}$ (ρ is density and γ is the adiabatic index) depend linearly on some spatial coordinates x_k /10/.

First we consider isentropic three-dimensional flows when the linear dependence on x_2 and x_3 has the form

$$Q=g(x_1),\quad u_1=g_1(x_1)$$
$$u_i=l_i(x_1)\,x_2+f_i(x_1)\,x_3+g_i(x_1),\quad i=2,3 \tag{1.1}$$

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The eight functions g, g_i, l_i, f_i satisfy the following set of ordinary differential equations /10/:

$$g_1 g_1' + \gamma(\gamma-1)^{-1} S_0 g' = 0, \quad g_1 l_i' + l_2 l_i + l_3 f_i = 0 \tag{1.2}$$
$$g_1 f_i' + f_2 l_i + f_3 f_i = 0, \quad g_1 g_i' + g_2 l_i + g_3 f_i = 0$$
$$g_1 g' + (\gamma-1) g (g_1' + l_2 + f_3) = 0$$

where $S_0 = \text{const}$ is the value of the entropy function in the equation of state $p = S_0 \rho^\gamma$, and p is the pressure. We find that Eqs.(1.2) are completely integrable in quadratures.

First we will obtain, in an obvious way, the three integrals (K_0 and A_i are arbitrary constants)

$$^1/_2 g_1^2 + \gamma(\gamma-1)^{-1} S_0 g = K_0, \quad g_i = A_2 l_i + A_3 f_i \tag{1.3}$$

The constants A_2, A_3 correspond to the transfer of the origin of coordinates along the axes x_2 and x_3, and we can assume without loss of generality that $A_2 = A_3 = 0$, i.e.

$$g_2 \equiv g_3 \equiv 0 \tag{1.4}$$

Furthermore, from the second and third equation of (1.2) (C_i are arbitrary constants) it follows that

$$f_2 = C_2 l_3, \quad l_2 - f_3 = C_3 l_3 \tag{1.5}$$

Eliminating from the second equation of (1.2) containing l_3' the functions l_2' and f_3 we obtain, using the last equation of (1.2) and (1.3), an equation for l_3 which, on integration, yields

$$l_3 = C_4 g_1 (\beta - g_1^2)^{1/(\gamma-1)}, \quad \beta = 2K_0, \quad C_4 = \text{const} \tag{1.6}$$

Substituting into the second equation of (1.2) for l_2' the representations for all functions l_i and f_i in terms of g_1, we obtain a second-order equation for $g_1(x_1)$. Substituting $^1/_2 g_1'^2 = y$ in this equation and taking g_1 as an independent variable, we obtain for $y(g_1)$ a first-order linear equation, which on integration yields

$$y = g_1(\beta - g_1^2)^{(2\gamma-1)/(\gamma-1)} [g_1^2(\gamma+1) - \beta(\gamma-1)]^{-2} (C^* - \tag{1.7}$$
$$2(\gamma-1)\alpha \int [x^2(\gamma+1) - \beta(\gamma-1)] (\beta - x^2)^{(2-\gamma)/(\gamma-1)} dx)$$
$$\alpha = (^1/_2 C_3^2 + C_2) C_4^2, \quad C^* = \text{const}$$

Computing in (1.7) the quadrature using the substitution $x(\beta - x^2)^{1/(\gamma-1)} = z$, we obtain the last integral containing arbitrary constants C^* and A_1

$$x_1 + A_1 = 2^{-1/2} \int [G(g_1)]^{-1/2} \left(-\gamma + \frac{\beta + g_1^2}{\beta - g_1^2}\right) [C^* + 2(\gamma-1)^2 \alpha G(g_1)]^{-1/2} dg_1, \quad G(g_1) = g_1(\beta - g_1^2)^{1/(\gamma-1)} \tag{1.8}$$

Relation (1.8) defines the function $g_1(x_1)$ implicitly, and the function l_2 has the form

$$l_2 = 2^{-1/2}(\gamma-1)^{-1} [G(g_1)]^{1/2} [C^* + 2(\gamma-1)^2 \alpha G(g_1)]^{1/2} \tag{1.9}$$

Thus we have obtained a general solution for Eqs.(1.2) depending on eight arbitrary constants $C_2, C_3, C_4, A_1, A_2, A_3, K_0, C^*$. Three of these constants A_1, A_2 and A_3 (A_1 defines the displacement along the x_1-axis) are not real, and we shall henceforth assume that $A_1 = A_2 = A_3 = 0$.

The class of solutions of (1.1) has the following geometrical property: the components u_2 and u_3 maintain a constant value along the stream lines, each of which represents a plane curve. Indeed, taking x_1 as the parameter along the stream lines we find from the second and third equation of (1.2), using the equations of the stream lines, that if $u_i = u_i^\circ = \text{const}\ (i = 2, 3)$ along some stream line, then this stream line lies in the plane

$$u_3^\circ x_2 - u_2^\circ x_3 = C^\circ = \text{const}$$

2. We shall consider the case when the class of solutions constructed defines asymmetric flows with x_1 as the axis of symmetry. Here it is sufficient to satisfy two conditions

$$l_2 f_2 + l_3 f_3 = 0, \quad l_2^2 + l_3^2 = f_2^2 + f_3^2 \tag{2.1}$$

obtain from (1.1), if we require that the radial velocity component $u_r = \sqrt{u_2^2 + u_3^2}$ should depend only on x_1 and $r = \sqrt{x_2^2 + x_3^2}$. Taking (1.5) into account, we obtain from (2.1) the following possibilities: 1) $l_3 = f_2 = 0$, $l_2 = f_3$; 2) $l_2 = f_3 = C_3 = 0$; 3) $C_2 = 1$, C_3 is arbitrary, and 4) $C_2 = -1$, $C_3 = 0$.

For case 1 we have $C_4 = 0$ in (1.6) and the stream lines lie in meridian planes passing through the x_1-axis. There, is no rotation of the stream about the x_1-axis. Case 2 leads to trivial solutions with $g = \text{const}$, $g_1 = \text{const}$. Case 3 for which $2l_2 - C_3 l_3 = 0$ leads, by virtue of the last equation of (1.2), to the same trivial possibility.

We shall consider case 4 in more detail

$$u_2 = l_2(x_1) x_2 - l_3(x_1) x_3, \quad u_3 = l_3(x_1) x_2 + l_2(x_1) x_3 \tag{2.2}$$
$$(u_r^2 = (l_2^2 + l_3^2) r^2)$$

We shall use the representations (2.2) to solve the following problem. Let the inlet cross section of the axisymmetric channel ($x_1 > 0$) be a circle, $x_2^2 + x_3^2 \leqslant R^2$, $x_1 = 0$, and let the following quantities be given:

$$g_1(0) = g_1^\circ > 0, \quad l_2(0) = l\cos\varphi_0, \quad l_3(0) = l\sin\varphi_0 \qquad (2.3)$$
$$\varphi_0 = \text{const}$$

Then Eqs.(2.2), which define at $x_1 = 0$ a field of transverse velocities of a gas flow twisted as a rigid body (such a twist is studied in /3/ for the case of a rotating block of solid fuel burning uniformly from one end). The parameter φ_0 determines the swirling intensity, and there is no swirling when $\varphi_0 = 0$. The form of the stream lines emerging from the points of the circumference q ($r = R$, $x_1 = 0$) and forming the walls of some nozzle at $x_1 > 0$, depends on the value of the parameter φ_0.

Let us investigate how the form of the nozzle and its thrust are affected by variations in φ_0 for various nozzle lengths L. We note that the flows in question are non-isoenergetic. The constant c_λ in the Bernoulli integral, for the stream lines passing at $x_1 = 0$ through the points of the circumference $r = \lambda$, is equal to

$$c_\lambda = {}^1/_2 l^2\lambda^2 + K_0 \quad (0 \leqslant \lambda \leqslant R)$$

We shall further assume that $K_0 = (\gamma - 1)^{-1}$ (this can be done without loss of generality by choosing a suitable system of units). From (1.6) we find

$$C_4 = l\sin\varphi_0 G_0^{-1}, \quad \alpha = -C_4^2, \quad G_0 = G(g_1^\circ) \qquad (2.4)$$

and (1.9) in this case yields

$$C^* = 2(\gamma - 1)^2 l^2 G_0^{-1} \qquad (2.5)$$

Thus we have found all the arbitrary constants. Next we determine the stream lines. We find that although the integral in (1.8) cannot, in general, be expressed in terms of elementary functions, the equations of the stream lines can be integrated explicitly using the quantity g_1 as a parameter along these lines.

Fixing on the circumference q the point

$$x_2^\circ = R\cos\varphi, \quad x_3^\circ = R\sin\varphi \qquad (2.6)$$

so that

$$u_2^\circ = Rl\cos(\varphi + \varphi_0), \quad u_3^\circ = Rl\sin(\varphi + \varphi_0) \qquad (2.7)$$

we use Eqs.(1.8),(2.4) and (2.5) to reduce the equations of the stream lines to the form

$$dx_k = -(\gamma - 1) B_k G^{-3/2} F^{-1/2} dG \qquad (2.8)$$
$$F = 1 - \sin^2\varphi_0 G_0^{-1} G, \quad B_k = u_k^\circ 2^{-1}(\gamma - 1)^{-1} l^{-1} G_0^{1/2}$$

Integrating (2.8), taking (2.6) and (2.7) into account, we reduce the parametric equations of the stream lines to the form

$$x_2 = R[\sin(\varphi + \varphi_0)\sin\varphi_0 + \cos(\varphi + \varphi_0) G_0^{1/2} G^{-1/2} F^{1/2}] \qquad (2.9)$$
$$x_3 = R[-\cos(\varphi + \varphi_0)\sin\varphi_0 + \sin(\varphi + \varphi_0) G_0^{1/2} G^{-1/2} F^{1/2}]$$

Analysis of the representations (2.9) yields directly the possible range of variation of g_1, namely $g_1 \in [g_-, g_+]$, $g_\pm = 2^{1/2}(\gamma \pm 1)^{-1/2}$. The value $g_1 = g_+$ corresponds to the point at which the stream lines reverse their direction, while the pressure and density both vanish at $g_1 = g_-$ (this happens when the stream lines recede to infinity). The integral in (1.8) will diverge, since ${}^1/_2(1 - 2\gamma)(\gamma - 1)^{-1} < -1$. The radius of the axisymmetric nozzle formed by the stream lines emerging from the circumference q is given by the expression

$$r = RG_0^{1/2} G^{-1/2} \qquad (2.10)$$

Clearly, when $g_1 \to g_-$, we have $r \to \infty$ and $x_1 \to \infty$, i.e. the nozzle walls diverge without limit. The following expression holds for the thrust $T_L(\varphi_0)$ of the nozzle when $x_1 \in [0, L]$ and $r \leqslant R_L(\varphi_0)$ ($R_L(\varphi_0) = r(L)$):

$$\frac{1}{2\pi} T_L(\varphi_0) = \int_0^R p_0 r\,dr + I_L(\varphi_0) - \frac{1}{2} p_L R_L^2(\varphi_0) \qquad (2.11)$$
$$I_L(\varphi_0) = \int_R^{R_L(\varphi_0)} pr\,dr$$

where p_0 is the pressure at the nozzle inlet $x_1 = 0$ and p_L is the back pressure at the cross section $x_1 = L$. The first and third term of (2.11) are independent of φ_0.

Consider the behaviour of the integral $I_L(\varphi_0)$. The function $G(g_1)$ (1.8) decreases monotonically ($\beta = 2(\gamma-1)^{-1}$) to zero when $g_1 \in [g_-, g_+]$. Let the second equation of (1.8) define the function $g_1 = g_1(G)$ and let $g_1 = g_1(L, \varphi_0)$, $G = G(L, \varphi_0)$ in the cross section $x_1 = L$. Then using (2.4) and (2.5) we obtain, from the integral (1.8), the following expression for $G(L, \varphi_0)$:

$$L = \frac{G_0^{1/2}}{2l} \int\limits_{G(L,\,\varphi_0)}^{G_0} g_1(G) G^{-3/2} F^{-1/2} dG \tag{2.12}$$

For fixed L it follows at once from (2.12) that the function $G(L, \varphi_0)$ increases monotonically for $\varphi \in [0, \pi/2]$ ($G(L, \varphi_0) < G_0$ and the integral (2.12) converges at the point $G = G_0$), while from (2.10) we find that $R_L(\varphi_0)$ is a monotonically decreasing function of φ_0. Using (2.10) we can write the integral $I_L(\varphi_0)$ in the form

$$I_L(\varphi_0) = I_0 \int\limits_{G(L,\,\varphi_0)}^{G_0} g_1^{-\gamma}(G) G^{\gamma-2} dG, \qquad I_0 = 2^{(1-2\gamma)/(\gamma-1)} S_0^{(\gamma-2)/(\gamma-1)} (\gamma-1)^{\gamma/(\gamma-1)} \gamma^{-1/(\gamma-1)} G_0 R^2 \tag{2.13}$$

When $L = \infty$, we have $G(\infty, \varphi_0) = 0$, and the integral (2.13) converges and is independent of φ_0 since $\gamma > 1$. When the length L is finite, the integral $I_L(\varphi_0)$ decreases monotonically as $\varphi_0 \in [0, \pi/2]$ increases. If $p_L = 0$ in (2.11) (no back pressure), then the thrust of the corresponding nozzles decreases for finite L as the swirling increases. Computations show that the reduction in thrust is insignificant.

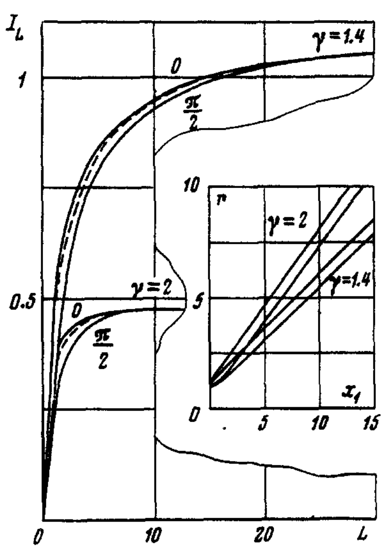

Fig. 1

Figure 1 shows graphs of $I_L(\varphi_0)$ (the dashed line corresponds to $\varphi_0 = \pi/4$) and the nozzle walls in the x_1, r-plane, for $\gamma = 1.4$ and $\gamma = 2$. The values of the remaining parameters are $R = 1$, $S_0 = 1$, $g_1^\circ = g_+ + 0.1$, $l = 1$. Note that the flow is supersonic everywhere within the nozzle. Indeed, we have the following expression for the speed of sound c:

$$c^2 = 1 - {}^1/_2(\gamma - 1) g_1^2$$

Then for $g_1 > g_+$ we have $|\mathbf{u}|^2 > c^2$. Now using (2.10) we obtain for large x_1 the following asymptotic estimates at the nozzle walls: $r = O(x_1)$, $p = O(x_1^{-2\gamma})$.

3. Consider three-dimensional flows with a linear dependence on a single coordinate

$$u_i = g_i(x_1, x_2), \quad i = 1, 2; \tag{3.1}$$
$$u_3 = f_3(x_1, x_2) x_3 + f_0(x_1, x_2)$$
$$Q = g(x_1, x_2), \quad S_0 = \text{const}$$

The function g_i, f_i and g satisfy the following set of equations:

$$g_1 \frac{\partial g_i}{\partial x_1} + g_2 \frac{\partial g_i}{\partial x_2} + \frac{\gamma}{\gamma-1} S_0 \frac{\partial g}{\partial x_i} = 0, \quad i = 1, 2; \tag{3.2}$$
$$g_1 \frac{\partial f_i}{\partial x_1} + g_2 \frac{\partial f_i}{\partial x_2} + f_3 f_i = 0, \quad i = 0, 3$$
$$g_1 \frac{\partial g}{\partial x_1} + g_2 \frac{\partial g}{\partial x_2} + (\gamma - 1) g \left(f_3 + \frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2} \right) = 0$$

We shall investigate the class of axisymmetric motions (3.1) with x_3 used as the axis of symmetry in the form

$$g_1 = a(\xi) x_1 - b(\xi) x_2, \quad g_2 = b(\xi) x_1 + a(\xi) x_2 \tag{3.3}$$
$$g = g(\xi), \quad f_i = f_i(\xi), \quad \xi = \sqrt{x_1^2 + x_2^2}$$

Substituting (3.3) into (3.2) we obtain the following set of equations for the functions a, b, g, f_i:

$$a(2b + b'\xi) = 0, \quad af_i'\xi + f_3 f_i = 0, \quad i = 0,3 \tag{3.4}$$
$$a^2 - b^2 + aa'\xi + \gamma(\gamma - 1)^{-1}S_0\xi^{-1}g' = 0$$
$$ag'\xi + (\gamma - 1)g(f_3 + 2a + a'\xi) = 0$$

The case $a \equiv 0$ (then we also have $f_3 \equiv 0$) corresponds to plane parallel flows, and we shall not consider it. The first two equations of (3.4) yield two simple integrals

$$b = b_0\xi^{-2}, \quad f_0 = C_0 f_3; \quad b_0 = \text{const}, \quad C_0 = \text{const} \tag{3.5}$$

We can write $C_0 = 0$ without loss of generality, since specifying the value of C_0 is equivalent to shifting the origin of coordinates along the x_3-axis. Using (3.5) we obtain another two integrals from the remaining equations of (3.4)

$$^1/_2\xi^2 a^2 + \gamma(\gamma - 1)^{-1}S_0 g + {}^1/_2 b_0^2\xi^{-2} = C_1 = \text{const} \tag{3.6}$$
$$g = C_2(a\xi^2)^{1-\gamma} f_3^{\gamma-1}, \quad C_2 = \text{const}$$

This reduces the problem of integrating the system (3.4) to that of integrating a single first-order equation

$$a\xi f_3' + f_3^2 = 0 \tag{3.7}$$

in which the function a must be expressed in terms of ξ and f_3 with help of (3.6). Integration in quadratures cannot be carried out when γ is arbitrary.

We note that in the case of the solutions of (3.1), Eq.(3.2) implies a property analogous to that established in Sect. 1, namely that the velocity component u_3 has a constant value along the stream lines. Unlike the previous case however, the stream lines are, generally speaking, three-dimensional curves.

The solutions of (3.3) correspond to non-isoenergetic vortex flows. We write the constant C_λ in the Bernoulli integral for the given stream line $(u_3 = u_{3\lambda} = \text{const})$, taking (3.6) into account, in the form

$$C_\lambda = C_1 + {}^1/_2 u_{3\lambda}^2 \tag{3.8}$$

In the plane case, when $u_3 \equiv 0$, the analogs of the solutions constructed represent potential spiral flows /11/ which are obtained by putting formally $f_3 \equiv 1$ in the second integral of (3.6). Addition of the component u_3 dependent on x_3 makes the flow (3.3) a vortex flow. From (3.6) it follows that, just as in the spiral flows, when $b_0 \neq 0$, we must exclude from the region of flow under consideration a neighbourhood of the axis $\xi = 0$ (otherwise g becomes negative), i.e. the flows should be constructed in annular axisymmetric channels.

4. Let us investigate the specific features of the solutions constructed in Sect. 3, regarding them as flows in semi-infinite annular channels $(x_3 \geqslant 1)$, the entry to which, at $x_3 = 1$, forms a ring $R_1 \leqslant \xi \leqslant R_2$. Using (3.6) and (3.7) we obtain the following expression for the flow rate P:

$$P = 2\pi\int_{R_1}^{R_2} \rho f_3\xi\, d\xi = 2\pi C_2^{1/(\gamma-1)}(f_3(R_1) - f_3(R_2)) \tag{4.1}$$

Therefore the total energy flux H will be given, taking (3.8) into account, by

$$H = 2\pi\int_{R_1}^{R_2} C_\lambda \rho f_3\xi\, d\xi = P\left[C_1 + \frac{1}{6}(f_3^2(R_1) + f_3(R_1)f_3(R_2) + f_3^2(R_2))\right] \tag{4.2}$$

Let us specify the parameter b_0 characterizing the energy of swirling of the gas flow, and the quantities R_1 and R_2. Then the set of equations (3.6) and (3.7), describing the class of flows in question, will contain unspecified constant C_1 and C_2, and a single initial condition for (3.7), such as e.g. $f_3(R_1) = f_3^\circ$. By choosing suitable values for these constants, we can solve a number of problems. In particular, having specified the quantities P, H and f_3°, we can consider relations (4.1) and (4.2) as a set of equations for determining C_1 and C_2, where the quantity $f_3(R_2)$, which depends explicitly on C_1, C_2, b_0, is determined implicitly after numerical integration of (3.7) for $\xi \in [R_1, R_2]$. In this manner we can, for example, study the influence of the degree of swirling on the thrust T in the cross section $x_3 = 1$

$$T(b_0) = 2\pi\int_{R_1}^{R_2} p(\xi)\xi\, d\xi, \quad p = S_0 g^{\gamma/(\gamma-1)} \tag{4.3}$$

From (3.4) and (3.6) we obtain the following representation for the derivative g':

$$g' = \frac{2(\gamma - 1)g[b_0^2 + a\xi^4(f_3 + a)]}{\xi[2C_1(\gamma - 1)\xi^2 - (\gamma + 1)\xi^4 a^2 - (\gamma - 1)b_0^2]} \tag{4.4}$$

From (4.4) we see that, depending on the values of the constants $C_1 > 0$, b_0, $a(R_2)$ (in place of the initial condition for $f_3(R_1)$ we can also specify $a(R_1)$), $g'(R_1)$, can take both positive and negative values. Thus we can have two types of flows: type A in which the density increases as the radius increases, $g'(R_1) > 0$, and type B in which the density decreases $g'(R_1) < 0$. From (3.6) it follows that the density cannot increase without limit as ξ increases $g \leqslant (\gamma - 1)\gamma^{-1}C_1S_0^{-1}$, and in the case of a B-type flow the flow density may fall to zero as $\xi \to \infty$. In the case of A-type flow the quantity f_3 tends very rapidly to zero with increasing ξ. If we have $g \approx g_0 = \text{const}$ for large ξ, then (3.6) yields $a\xi^2 f_3^{-1} \approx \nu_0 = \text{const}$ and (3.7) then yields $f_3 = 0\ [s(\xi)]$, $a\xi = O[\xi^{-1}s(\xi)]$, $s(\xi) = \exp(-{}^1/_2\nu_0\xi^2)$. Thus the longitudinal component with respect to the axis of rotation, and the radial component of the velocity vector both decrease rapidly as the radius increases so that the flow is transformed into a plane vortex. For B-type flows the quantity f_3 decreases as ξ increases much more slowly. If for large $\xi\ g \to 0$ and $a\xi = \nu_1 = \text{const} > 0$, then from (3.6) we have $f_3 = O(\xi^{-1})$.

Let us present some results of numerical computations of A- and B-type flows. The following problem was solved. The values $P, H, f_3(R_1)$ and $f_3(R_2)$ in (4.1) and (4.2) were fixed and used to obtain the constants C_1 and C_2, while the quantity R_2 representing the outer radius of the annulus was found by numerical integration of (3.7), with the initial value at $R = R_1$ ensuring that at $\xi = R_2$ the function f_3 would take a given value $f_3(R_2)$. The parameter b_0 was varied.

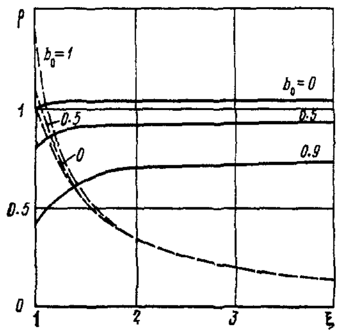

Fig.2

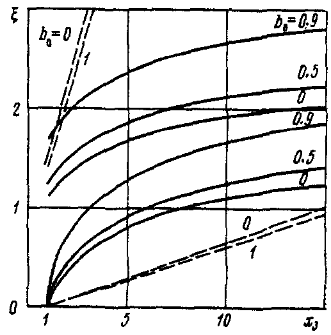

Fig.3

Figure 2 shows graphs of the density for various values of b_0 ($R_1 = 1$; $f_3(1) = 1$; $f_3(R_2) = 0.1$; $\gamma = 1.4$). For A-type flows (solid lines) $P = 1.79$; $H = 6.68$, and for B-type flows (dashed lines) $P = 13.85$; $H = 92.6$.

Figure 3 shows the walls of annular channels for both types of flow (the scale should be increased fifty-fold for case B). For A-type flows the quantity $T(b_0)$ decreases as b_0 increases: $T(0) = 4.63$; $T(0.5) = 4.36$; $T(0.9) = 3.75$, and for B-type flows it increases: $T(0) = 20.42$; $T(0.5) = 20.59$; $T(1) = 21.49$.

Unlike the supersonic solutions of Sect. 2, the flows discussed here can be of mixed type, with both supersonic and subsonic zones present. The equation of sonic surface has the form

$$f_3^2x_3^2 + a^2\xi^2 + b_0^2\xi^{-2} = \gamma S_0 g(\xi)$$

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ON THE PROPAGATION OF NON-STEADY PERTURBATIONS IN A BOUNDARY LAYER WITH SELFINDUCED PRESSURE*

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The problem of supersonic flow past a flat plate bearing a triangular vibrator which begins to execute harmonic oscillations in the unperturbed boundary layer, is studied. Both the plate and vibrator are assumed to be thermally insulated. The size of the vibrator and the frequency of its oscillations are such that the flow can be described by the equations for a boundary layer with selfinduced pressure. The oscillation amplitude is assumed to be small, and this enables the equations to be linearized. The linear formulation is used to study the problem where the pressure reaches a steady-state periodic mode. The problem of the vibrator is used to solve the problem of the propagation of non-steady perturbations both upstream and downstream.

1. **Formulation of the problem and its formal solution.** Consider the flow of an ideal gas past a thermally insulated body representing a flat plate with an irregularity positioned at some distance from the ends, and of which changes its form with time. We shall assume that the parameters of the incoming unperturbed flow (U_∞^* is the velocity, p_∞^* is the pressure and ρ_∞^* is the density) determine the Mach number $M_\infty > 1$ (here and henceforth the subscript ∞ refers to the parameters in the unperturbed flow). We assume that the dependence of the first coefficient of viscosity on temperature T^* is linear $\lambda_1^*/\lambda_{1\infty}^* = CT'$ $(T' = T^*/T_\infty^*)$, and the Prandtl number is equal to unity. The distance from the leading edge of the plate to the irregularity will be denoted by L^*, and in place of the reciprocal of the Reynolds number we shall use the small parameter $\varepsilon = \mathrm{Re}_1^{-1/8}$ $(\mathrm{Re}_1 = \rho_\infty^* U_\infty^* L^*/\lambda_{1\infty}^*)$.

Let us choose the longitudinal dimension of the irregularity $O(L^*\varepsilon^3)$, the transverse dimension $O(L^*\varepsilon^5)$, and the characteristic time of variation in the form of the irregularity $O(L^*\varepsilon^2/U_\infty^*)$. To describe the motion in the neighbourhood of such an irregularity it is convenient to separate three characteristic regions /1, 2/: the upper region of the supersonic inviscid flow $(y_1^* = O(L^*\varepsilon^3))$, the intermediate region of the conventional boundary layer $(y_2^* = O(L^*\varepsilon^4))$ and the lower region of the boundary layer with selfinduced pressure. The principal difficulties that arise in such a scheme are connected with constructing a solution in the lower region, and investigation of this region is the purpose of this paper.

Let us introduce the dependent and independent dimensionless variables used in /3, 4/ and denote by x and y the Cartesian coordinate axes, with x directed along the plate, u and v the velocity vector components along the x and y axes, p the pressure and ρ the density. By requiring that the conditions of merging with the conventional boundary layer hold as $x \to -\infty$ and $y \to \infty$, we obtain, from the Navier-Stokes equations for the pincipal terms of the expansion as $\varepsilon \to 0$, a set of equations for the unsteady boundary layer with selfinduced pressure /3-5/. We shall describe the irregularity (Fig.1) in the form

$$y_w = \sigma f(t, x), \quad \sigma \ll 1 \tag{1.1}$$

and demand that the conditions of adhesion hold on the plate and the irregularity. This gives the following relations for the approximation $(\varepsilon \to 0)$ used:

$$u(t, x, y_w) = 0, \quad v(t, x, y_w) = \sigma \partial f/\partial t \tag{1.2}$$

Fig.1

*Prikl.Matem.Mekhan., 47, 5, 762-770, 1983.